Quantum Teleportation and Quantum State Discrimination

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1. Background

1.1 Entanglement

Entanglement is a particular form of genuinely quantum correlation between quantum systems. Since entanglement cannot be created "from scratch" using local operations and classical communication only, it is a valuable resource in quantum communication.

1.2 Noise & Quantum Channels

Noise in a quantum system can be understood as an unwanted interaction with the environment that hinders quantum communication and computation. A **quantum channel** is a mathematical model for this noise, which we can think of as a noisy communication link.

1.3 Quantum Teleportation

Quantum teleportation is a task that implements a quantum channel through entanglement and classical communication. Alice measures the system to be teleported and one half of an entangled state shared with Bob. Sending the measurement outcome to Bob allows him to post-process his share of the entanglement in order to achieve teleportation.

1.4 Quantum State Discrimination

Quantum state discrimination is a related problem where we are given a quantum system in a state ρ_i with probability p_i for $i = 1, \dots, N$, and the goal is to determine the true state. We aim to maximize the success probability $P_{\rm succ} = \sum_{i=1}^N p_i \operatorname{tr}(\Pi_i \rho_i)$ of correctly identifying the state, where $\{\Pi_i\}_{i=1}^N$ is a measurement satisfying

$$\Pi_i \geq 0 \; \forall i \in 1, .., N \quad \text{and} \quad \sum_{i=1}^N \Pi_i = \mathbb{I}.$$

This optimization problem is an example of a so-called semi-definite program.

2. Problem Statement

We consider a general teleportation protocol consisting of the following data and steps:

- A, B, C are quantum systems of arbitrary dimension. AC is with Alice, B with Bob.
- Alice and Bob share an entangled quantum state ρ_{AB}.
- Protocol:
  1. Alice performs a measurement $\Pi = \{\Pi^i_{AC}\}_{i=1}^N$ on her systems.
  2. Alice communicates the measurement outcome $i \in [N]$ to Bob.
  3. Bob applies a decoding operation (quantum channel) $\mathcal{D}_i \colon B \to C$.

The resulting teleportation channel $\Lambda\colon C \to C$ has entanglement fidelity

$$F = \frac{1}{|C|^2} \sum_{i=1}^N \operatorname{tr}\left(\Pi^i_{AC} \omega^i_{AC}\right), \tag{2.1}$$

where we defined the states $\omega^i_{AC} = (\mathrm{id}_A \otimes \mathcal{D}_i)(\rho_{AB})$ [3].

To get the optimal entanglement fidelity for a fixed shared state ρ_{AB}, we can optimize (2.1) over the measurement Π for a fixed set of decoding operations $\{\mathcal{D}_i\}_{i=1}^N$:

$$\begin{aligned} \text{maximize: } & \frac{1}{|C|^2} \sum_{i=1}^N \operatorname{tr}\left(\Pi^i_{AC} \omega^i_{AC}\right) \\ \text{subject to: } & \Pi^i_{AC} \geq 0 \text{ for all } i = 1, \dots, N, \\ & \sum_{i=1}^N \Pi^i_{AC} = \mathbb{1}_{AC}. \end{aligned} \tag{2.2}$$

Alternatively, we can fix Π and instead optimize the decoding operations $\{\mathcal{D}_i\}_{i=1}^N$:

$$\begin{aligned} \text{maximize: } & \frac{1}{|C|^2} \sum_{i=1}^N \operatorname{tr}\left(\Pi^i_{AC} \omega^i_{AC}\right) \\ \text{subject to: } & \text{for all } i \in [N] \text{ the Choi operator } \tau^i_{BC} \text{ of } \mathcal{D}_i \colon B \to C \text{ satisfies} \\ & \tau^i_{BC} \geq 0 \\ & \operatorname{tr}_C \tau^i_{BC} = \mathbb{1}_B. \end{aligned} \tag{2.3}$$

Both optimizations are semidefinite programs and can be implemented using e.g. `CVX` in `MATLAB` or `python`. We can also try to optimize over both Π and $\{\mathcal{D}_i\}_{i=1}^N$ at the same time using more general optimization methods. Note that this is a bilinear problem and hence *hard to solve in general.*

Using the Choi isomorphism, the SDP in (2.4) can be made explicit to give the following new SDP:

$$\begin{aligned} \text{maximize: } & \frac{1}{|C|^2} \sum_{i=1}^N \operatorname{tr}\left[\Pi^i_{AC} \operatorname{tr}_B\left[\left(\mathbb{1}_A \otimes \tau^i_{BC}\right)\left(\rho^{T_B}_{AB} \otimes \mathbb{1}_C\right)\right]\right] \\ & = \frac{1}{|C|^2} \sum_{i=1}^N \operatorname{tr}\left[\left(\Pi^i_{AC} \otimes \mathbb{1}_B\right)\left(\mathbb{1}_A \otimes \tau^i_{BC}\right)\left(\rho^{T_B}_{AB} \otimes \mathbb{1}_C\right)\right] \\ \text{subject to: } & \tau^i_{BC} \geq 0 \quad \text{and} \quad \operatorname{tr}_C \tau^i_{BC} = \mathbb{1}_B \text{ for all } i \in [N]. \end{aligned} \tag{2.4}$$

3. Results

We focus on the simple situation where A, B, C are qubits. We assume that Alice and Bob share the following two-qubit state ([1]):

$$\rho_{AB} = \begin{pmatrix} a & 0 & 0 & y \\ 0 & b & x & 0 \\ 0 & x & c & 0 \\ y & 0 & 0 & d \end{pmatrix} \tag{3.5}$$

where $a, b, c, d, x, y \in \mathbb{R}$ are such that $\rho_{AB} \geq 0$ and $\operatorname{tr} \rho_{AB} = 1$.

We use the following values from [1]:

$$a = 0 \qquad b = \frac{3 - 2\sqrt{2}}{2} \qquad c = \frac{1}{2} \qquad d = \sqrt{2} - 1 \qquad x = \frac{1 - \sqrt{2}}{2} \qquad y = 0 \tag{3.6}$$

In particular, b, c, x are constant in the left panel of Fig. 1, and a, d, y are constant in the right panel.

We now solve the SDPs in (2.2) and (2.4) in order to obtain teleportation protocols with fidelities above the so-called "classical fidelity" of $1/2$, which can be achieved without any entanglement between A and B.

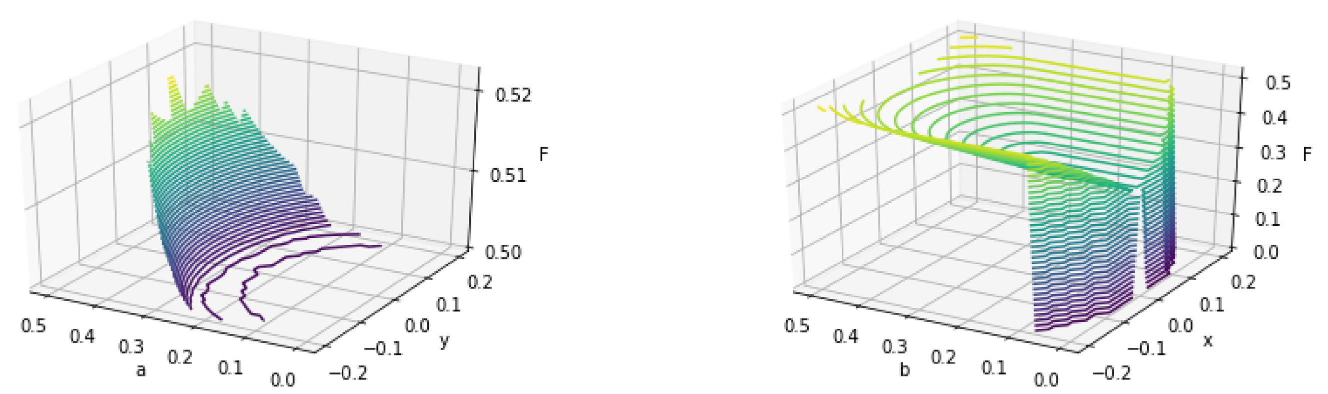

Figure 1: *Two 3-dimensional subspaces of* $\{x, y, a, b, c, d\} \to F$*, showing maximum fidelities above 0.5. We optimize over measurements with the Pauli corrections from the standard teleportation protocol [2] as the decoding operations.*

We also take a look at the quantum state $\tilde{\rho}_{AB} = (\mathrm{id}_A \otimes \mathcal{A}_p)(\rho_{AB})$ where $\mathcal{A}_p(\cdot)$ is the *amplitude damping channel* with Kraus operators:

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \qquad E_2 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}. \tag{3.7}$$

This is a noisy version of the initially shared quantum state in (3.5). Interestingly, the paper [1] found that for the values of ρ_{AB} as in (3.6), the fidelity equals $1/2$, while amplitude damping noise improves the teleportation fidelity, as shown in Fig. 2.

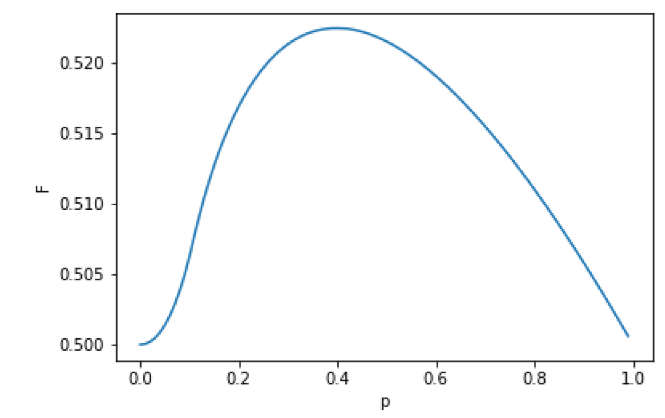

Figure 2: *We apply the amplitude dampening channel to the B subsystem and again optimize over measurements with the Pauli corrections from [2] as the decoding operations. We plot the obtained fidelity F using the values from* (3.6) *as we vary p, showing a maximum fidelity of $F = 0.5224$ at $p^* = 0.397$.*

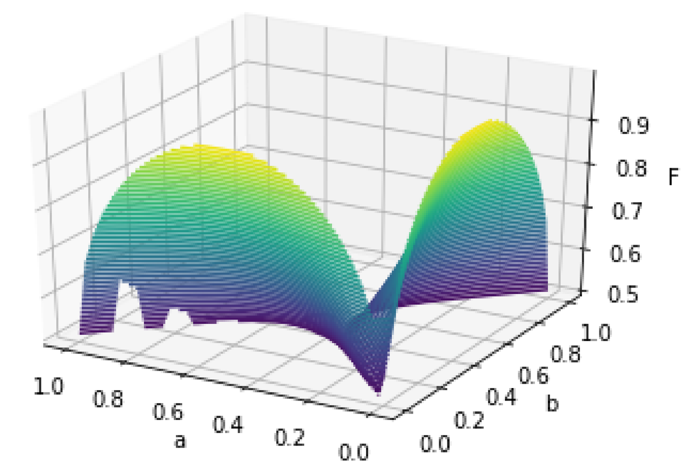

Figure 3: *We realize our best protocol and best fidelity when using the Bell measurement from [2] and optimizing over decoders via the SDP* (2.3)*. In this chart, we assume $\frac{a}{b} = \frac{d}{c}$ and $y = \sqrt{ad}, x = \sqrt{bc}$.*

4. Conclusion

We optimized teleportation protocols using a specific entangled resource state, and found that decoding operations different than the usual Pauli corrections can increase the teleportation fidelity.

In particular, we have reproduced the result from [1] that fidelity can be increased if the state is made more noisy, which may seem counter-intuitive.

In the future, we can look for optimized teleportation protocols when considering multipartite entangled states or locality constraints on the measurement operators.

References

[1] Badziag, M. Horodecki, P. Horodecki, and R. Horodecki. "Local environment can enhance fidelity of quantum teleportation". Phys. Rev. A 62 (1 June 2000), p. 012311. arXiv: quant-ph/9912098. [2] Bennett et al. "Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels". Phys. Rev. Lett. 70, 1895 (1993). [3] Chitambar, Leditzky. Manuscript in preparation (2022).

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